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Unambiguous comparison of the states of multiple quantum systems

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Abstract

We consider N quantum systems initially prepared in pure states and address the problem of unambiguously comparing them. One may ask whether or not all N systems are in the same state. Alternatively, one may ask whether or not the states of all N systems are different. We investigate the possibility of unambiguously obtaining this kind of information. It is found that some unambiguous comparison tasks are possible only when certain linear independence conditions are satisfied. We also obtain measurement strategies for certain comparison tasks which are optimal under a broad range of circumstances, in particular when the states are completely unknown. Such strategies, which we call universal comparison strategies, are found to have intriguing connections with the problem of quantifying the distinguishability of a set of quantum states and also with unresolved conjectures in linear algebra. We finally investigate a potential generalization of unambiguous state comparison, which we term unambiguous overlap filtering.

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1. Introduction

In classical physics, the state of a physical system is the information which specifies the values of its dynamical variables. There are no fundamental limitations on the precision with which these variables can be measured simultaneously. So, in classical physics the state of a physical system can, in principle, be measured with arbitrarily high accuracy. This contrasts with the situation in quantum mechanics. The state of a quantum mechanical system, is, in the best case scenario, represented by a normalized vector in the system's associated Hilbert space. The only state vectors that can be completely distinguished from one another are vectors

belonging to a known, orthogonal set. As a consequence, it is impossible to measure the state of a quantum system if we do not have prior information specifying an orthogonal set to which it belongs.

This fact has inspired a great deal of research relating to the limitations imposed by the quantum formalism on our ability to obtain information about the state of a quantum system. In the investigation of this matter, different approaches relating to correspondingly distinct scenarios have been developed [1]. For example, in quantum state discrimination, the aim is to determine the state as well as possible, given that we know that the system has been prepared in one of a finite number of possibly non-orthogonal states, and with a knowledge of their *a priori* probabilities. In quantum state estimation, we do not have this information and we aim to determine, as well as possible, a completely unknown state, although perhaps with multiple copies of this state.

In this paper, we investigate the possibility of obtaining a different kind of information about quantum states. Here, we are given N quantum systems, which are physically of the same kind and have identical Hilbert spaces. Each of the N systems has the same set of possible states which, throughout, we assume to be pure. Our aim is to determine whether or not the actual states of these systems are all identical, or all different. This kind of procedure is known as quantum state comparison.

There has recently been a substantial amount of work done in relation to this matter. The first investigation focused on comparing the states of just two quantum systems [2]. Subsequently, the possibility of comparing unitary operators was proposed and explored [3]. Also, the links between comparison of pure quantum states and discrimination between mixed quantum states are described in [2, 4]. Two applications in quantum communications where state comparison plays an important role are quantum fingerprinting [5] and quantum digital signatures [6]. Quantum state comparison has also been proposed as a technique for quantum computer stabilization [7].

More recently, we have explored numerous aspects of the problem of comparing the states of many quantum systems [8]. This paper is in many ways a companion paper to this work. In [8], different kinds of comparison strategy were examined for several sets of possible states, and techniques for experimentally implementing these strategies were proposed. The aim of this paper is to present a detailed analysis of the problem of unambiguous state comparison for multiple quantum systems with multiple possible states. We also give rigorous proofs of some of the assertions made in [8].

In unambiguous state comparison, the aim is to establish the identicality of, or differences among the states of many quantum systems, with zero probability of error. This cannot be achieved in a completely reliable way if the states are non-orthogonal. We must, in general, allow for a non-zero probability that our comparison attempt will yield an inconclusive result.

We begin in section 2 with a brief review of generalized quantum measurements, which we shall use extensively throughout this paper. For a more detailed treatment of such matters, see [9]. Our investigation of unambiguous state comparison commences in section 3. Here, we examine the problem of confirming, unambiguously, that all N particles have been prepared in the same state, a task that we term identicality confirmation. We show that if this is to succeed with non-zero success probability for each element of a discrete set of N-particle states, then this set, or equivalently, the set of possible single-particle states, must be linearly independent.

On the other hand, if we wish to unambiguously confirm that the N particles are not all prepared in the same state, then no such restriction applies. In section 4 we describe a universal measurement strategy for unambiguous confirmation that not all N particles have been prepared in the same state. This strategy is optimal under a broad range of circumstances, in particular when the single-particle states are uniformly distributed.

Alternatively, we might wish to know when the states of the N particles are all different from each other. We address this problem in section 5. Here we show that the possible single-particle states must obey a weaker linear independence condition than that in section 3. We find that this linear independence condition is always sufficient and obtain a universal measurement for confirming that all N particles are in different states. This measurement succeeds with non-zero probability whenever this linear independence condition is satisfied. It is also shown to be optimal when all N-particle product states are possible.

In section 6, we examine the above universal unambiguous comparison strategies in greater detail. In particular, we investigate the possibility that their associated success probabilities are suitable measures of the distinguishability of the states of the individual particles, considered as possible states of a single particle. For this to be the case, these probabilities must satisfy certain conditions. In particular, they must be non-increasing under all deterministic quantum operations which transform these pure states into another set of pure states. We find that this is indeed the case for the second of our universal comparison strategies. The question of whether or not it is for the first strategy is found to be equivalent to a certain, currently unresolved conjecture in linear algebra.

The final topic we discuss, in section 7, is a procedure we term unambiguous overlap filtering. In unambiguous state comparison, our aim is to determine, with zero probability of error, whether or not a number of quantum systems have been prepared in the same pure state. This is equivalent to determining whether or not they have mutual overlap equal to 1. A potential generalization of this procedure is the unambiguous determination of whether or not they have overlap equal to some fixed $\omega \in [0, 1]$. We concentrate on the states of just a pair of systems. We find that, when these states are pure but otherwise arbitrary and unknown, then it is impossible, with any non-zero probability, and for any value of $\omega \in [0, 1]$, to unambiguously confirm that they have overlap equal to ω. We also find that it is impossible to unambiguously confirm, with non-zero probability, that their overlap is not equal to ω unless $\omega = 1$, which corresponds to difference detection. Therefore, the only possible kind of unambiguous overlap filtering for this set of states is unambiguous difference detection. We conclude in section 8 with a general discussion of our results.

2. State comparison and generalized measurements

In this paper, we will investigate problems relating to quantum state comparison. We will be focusing on questions such as 'are the states of N quantum systems all identical?' and 'are the states of N quantum systems all different?'. If we know the states of the individual systems, then questions such as these can be readily answered by comparing explicit expressions for these states. If, however, we do not know these states in advance, then to obtain the required information about our N-particle system, we must perform measurements on it. In order to assess the scope and limitations of the quantum measurement process with respect to such problems, we should consider the most general class of measurements that are physically possible on our N-particle system. In introductory texts on quantum mechanics, e.g. [10], the measurements which are most commonly considered are von Neumann measurements on a single quantum system. There are two possible extensions of this class of measurements, which, when permitted, make accessible to us the full range of physically possible measurement procedures on an N-particle system. The first extension involves lifting the restriction to measurements on single particles, and to allow for the possibility of collective measurements on several, potentially all N particles. In numerous contexts in quantum information theory, it has been found that such collective measurements can be used to achieve tasks that are impossible using single-particle measurements. Perhaps the best-known example is the Bell

measurement, which is an essential component of many important quantum communications protocols, such as teleportation [11] and dense coding [12].

The second extension is the lifting of the restriction to von Neumann measurements. A much broader class of measurements on quantum systems are possible, which are appropriately known as generalized measurements. The statistical properties of such a measurement are described in terms of a set of positive operators[4] known as positive, operator-valued measure (POVM) elements. If we have a generalized measurement (henceforth simply measurement) with K possible outcomes, indexed by $k = 1, \ldots, K$, then associated with the kth outcome is a POVM element E_k. These operators act on the Hilbert space of the total system under investigation and we denote this by $\mathcal{H}_{\mathrm{tot}}$.

The main purpose served by the POVM elements is to provide the probability distribution for the measurement outcomes for each initial state. Given that the initial state is described by a density operator ρ acting on $\mathcal{H}_{\mathrm{tot}}$, the probability of obtaining outcome k for this measurement will be denoted by $P_k(\rho)$. Whether or not a generalized measurement is possible with specified probabilities $P_k(\rho)$ depends on whether or not the following two necessary and sufficient conditions are satisfied. Firstly, there must exist a set of K positive operators E_k, which will be the POVM elements of the measurement, such that the probabilities have the form

$$P_k(\rho) = \mathrm{Tr}(\rho E_k). \tag{2.1}$$

When the system is initially in a pure state $\rho = |\Psi\rangle\langle\Psi|$, this expression may be written as

$$P_k(\Psi) = \langle\Psi|E_k|\Psi\rangle. \tag{2.2}$$

Secondly, the K POVM elements must form a resolution of the identity,

$$\sum_{k=1}^{K} E_k = 1_{\mathrm{tot}}, \tag{2.3}$$

where 1_{tot} is the identity operator on the Hilbert space $\mathcal{H}_{\mathrm{tot}}$. This expression is equivalent to the normalization of the outcome probability distribution for the all initial states ρ,

$$\sum_{k=1}^{K} P_k(\rho) = 1. \tag{2.4}$$

A collective, generalized measurement is the most general measurement procedure that can be carried out on a multiparticle quantum system. In our subsequent discussion of quantum state comparison, we will seek to obtain results of maximum possible generality. With this in mind, the properties of collective, generalized measurements, in particular, what can and what cannot be achieved with them, will be of considerable importance.

3. Unambiguous identicality confirmation for linearly independent states

Consider the following situation. We have N particles indexed by the label $j = 1, \ldots, N$. Associated with each of them is a copy of the finite-dimensional Hilbert space $\mathcal{H}$. We shall denote the dimensionality of this space by $D(\mathcal{H})$. The Hilbert space of the entire N-particle system is $\mathcal{H}_{\mathrm{tot}} = \mathcal{H}^{\otimes N}$. Each particle has been prepared in an element of the set of M pure states $\{|\psi_\mu\rangle\}$, with $\mu = 1, \ldots, M$. All the states in this set are different from each other. Throughout this paper, two pure states, say $|\psi_1\rangle$ and $|\psi_2\rangle$, will be considered identical if

[4] Throughout this paper, when we speak of positive operators and matrices, we mean operators and matrices that are positive semidefinite rather than strictly positive definite.

$|\langle\psi_1|\psi_2\rangle| = 1$ and different otherwise. We denote the actual state of particle j by $|\psi_{\mu_j}\rangle$. We may write the state of the N-particle system in tensor product form,

$$|\Psi_{\mu_1,\ldots,\mu_N}\rangle = \bigotimes_{j=1}^{N} |\psi_{\mu_j}\rangle. \tag{3.1}$$

Our N-particle system has been prepared in one of the M^N states $|\Psi_{\mu_1,\ldots,\mu_N}\rangle$. We would like to find a measurement that enables us to unambiguously confirm when all N particles have been prepared in the same state.

This task can sometimes be achieved by unambiguous state discrimination. It is known that one can determine unambiguously in which element of a finite set of pure states a quantum system has been prepared, with non-zero probability for each element, if and only if this set is linearly independent [13]. This procedure could clearly be applied to the problem at hand for linearly independent $|\Psi_{\mu_1,\ldots,\mu_N}\rangle$. If this linear independence condition is satisfied, then we could unambiguously determine, with non-zero probability, in which of the states $|\Psi_{\mu_1,\ldots,\mu_N}\rangle$ our N-particle system has been prepared. When this unambiguous discrimination attempt succeeds, it will give the values of the μ_j. We need then only compare the values of the μ_j and determine whether or not they are all equal to accomplish the state comparison task in question. So, we see that linear independence of the $|\Psi_{\mu_1,\ldots,\mu_N}\rangle$ is a sufficient condition for being able to unambiguously confirm that all N particles have been prepared in the same state.

Furthermore, one can easily show that the linear independence of the possible N-particle states $|\Psi_{\mu_1,\ldots,\mu_N}\rangle$ is equivalent to that of the possible single-particle states $|\psi_\mu\rangle$. One way to prove this is to make use of the fact that, for any density operator ρ which is a statistical mixture of $M(\rho)$ pure states, these states are linearly independent if and only if $\mathrm{rank}(\rho) = M(\rho)$. With this in mind, consider the density operators

$$\rho_1 = \frac{1}{M}\sum_{\mu=1}^{M} |\psi_\mu\rangle\langle\psi_\mu|, \tag{3.2}$$

$$\rho_N = \frac{1}{M^N}\sum_{\mu_1,\ldots,\mu_N=1}^{M} |\Psi_{\mu_1,\ldots,\mu_N}\rangle\langle\Psi_{\mu_1,\ldots,\mu_N}| = \rho_1^{\otimes N}. \tag{3.3}$$

Clearly, $M(\rho_1) = M$ and $M(\rho_N) = M^N$. From the second part of equation (3.3), we see that $\mathrm{rank}(\rho_N) = [\mathrm{rank}(\rho_1)]^N$. From these observations, we find that $\mathrm{rank}(\rho_N) = M(\rho_N)$ if and only if $\mathrm{rank}(\rho_1) = M(\rho_1)$ and this completes the proof. We can then conclude that a sufficient condition for being able to unambiguously confirm, with non-zero probability, identicality of the states of the N particles is that the set of possible single-particle states is linearly independent.

We shall see here that linear independence of the possible single-particle states or, equivalently, that of the possible N-particle states, is also a necessary condition for unambiguous confirmation that all N particles are in the same state. To see this, let us formulate the problem as a generalized measurement in the way described in section 2. The measurement will have three potential outcomes: 'same', 'different' and '?'. The first pair of outcomes signal that all N particles have been prepared in the same state, and that they have not all been prepared in the same state, respectively. The third possible outcome, '?', is the inconclusive result. On obtaining this, we are none the wiser about whether or not the states of all N particles are identical. Corresponding to these possible outcomes are the POVM elements E_{same}, E_{diff} and $E_?$. These are positive operators acting on $\mathcal{H}_{\mathrm{tot}}$ which satisfy the resolution of the identity in equation (2.3).

The measurement outcome that we are particularly interested in is that which signals that all N particles have been prepared in the same state. We denote by $P_{\text{same}}\big(\Psi_{\mu_1,\ldots,\mu_N}\big)$ the probability of obtaining this 'same' result for the initial state $\big|\Psi_{\mu_1,\ldots,\mu_N}\big\rangle$. From equation (2.2), we see that this is given by

$$P_{\text{same}}\big(\Psi_{\mu_1,\ldots,\mu_N}\big) = \big\langle\Psi_{\mu_1,\ldots,\mu_N}\big|E_{\text{same}}\big|\Psi_{\mu_1,\ldots,\mu_N}\big\rangle. \tag{3.4}$$

For the measurement result to be unambiguous and to succeed with non-zero probability whenever all N particles have been prepared in the same state, we require that $P_{\text{same}}\big(\Psi_{\mu_1,\ldots,\mu_N}\big) > 0$ if and only if the μ_j are all equal.

To prove that this requirement can only be satisfied if the single-particle states are linearly independent, let us consider the N-particle states of the form $\big|\Psi_{\mu_1\mu\ldots\mu}\big\rangle$. Here, particle 1 is in the state $\big|\psi_{\mu_1}\big\rangle$ and the remaining particles are all in the state $|\psi_\mu\rangle$. It is clear from the above discussion that the result 'same' can be obtained with non-zero probability if and only if $\mu_1 = \mu$. We may then write

$$P_{\text{same}}\big(\Psi_{\mu_1\mu\ldots\mu}\big) = p_\mu\delta_{\mu_1\mu} \tag{3.5}$$

for some $p_\mu > 0$. Let us now attempt to write the set of possible states of particle 1 as superpositions of each other,

$$\big|\psi_{\mu_1}\big\rangle = \sum_{\nu=1}^{M} f_{\mu_1\nu}|\psi_\nu\rangle. \tag{3.6}$$

The $\big|\psi_{\mu_1}\big\rangle$ are linearly independent if and only if the only possible values of the coefficients $f_{\mu\nu}$ are $f_{\mu\nu} = \delta_{\mu\nu}$. In fact, this is equivalent to $|f_{\mu\nu}| = \delta_{\mu\nu}$, a formulation of this condition that will be more convenient.

Applying equation (3.6) to the N-particle state $\big|\Psi_{\mu_1\mu\ldots\mu}\big\rangle$, we obtain

$$\big|\Psi_{\mu_1\mu\ldots\mu}\big\rangle = \sum_{\nu=1}^{M} f_{\mu_1\nu}|\Psi_{\nu\mu\ldots\mu}\rangle \tag{3.7}$$

which gives us

$$\langle\Psi_{\mu_1\mu\ldots\mu}|E_{\text{same}}\big|\Psi_{\mu_1\mu\ldots\mu}\big\rangle = \sum_{\nu,\nu'=1}^{M} f^*_{\mu_1\nu'}f_{\mu_1\nu}\langle\Psi_{\nu'\mu\ldots\mu}|E_{\text{same}}|\Psi_{\nu\mu\ldots\mu}\rangle. \tag{3.8}$$

From the Cauchy–Schwarz inequality, we see that

$$|\langle\Psi_{\nu'\mu\ldots\mu}|E_{\text{same}}|\Psi_{\nu\mu\ldots\mu}\rangle|^2 \leqslant \langle\Psi_{\nu'\mu\ldots\mu}|E_{\text{same}}|\Psi_{\nu'\mu\ldots\mu}\rangle\langle\Psi_{\nu\mu\ldots\mu}|E_{\text{same}}|\Psi_{\nu\mu\ldots\mu}\rangle = p_{\nu'}p_\nu\delta_{\nu'\mu}\delta_{\nu\mu} \tag{3.9}$$

from which we obtain

$$\langle\Psi_{\nu'\mu\ldots\mu}|E_{\text{same}}|\Psi_{\nu\mu\ldots\mu}\rangle = p_\mu\delta_{\nu'\mu}\delta_{\nu\mu}. \tag{3.10}$$

Substitution of this into equation (3.8) and making use of equations (3.4) and (3.5) gives

$$|f_{\mu_1\mu}|^2 p_\mu = p_\mu\delta_{\mu_1\mu}. \tag{3.11}$$

All the probabilities p_μ are greater than zero, so we finally obtain

$$|f_{\mu_1\mu}| = \delta_{\mu_1\mu} \tag{3.12}$$

showing that the states must be linearly independent. This completes the proof.

So, we see that a necessary and sufficient condition for confirming identicality of the states of all N particles is a condition which is both necessary and sufficient for unambiguously

discriminating among all possible N-particle states, or equivalently, all possible single-particle states.

Unambiguous state comparison may be viewed as unambiguous discrimination between two subsets of the entire set of possible N-particle pure states. In one subset, all N particles are prepared in the same state, and in the other, they are not. The fact that unambiguous discrimination between these two subsets is possible if and only if we can unambiguously discriminate among all possible N-particle states is interesting and non-trivial. Even though unambiguous discrimination among all individual states in a set is possible only if the states are linearly independent, Sun *et al* [14] showed that this condition is not always necessary if we only wish to unambiguously discriminate among subsets instead. However, it is necessary here.

4. Universal unambiguous detection of at least one difference

4.1. Universal measurement strategy

The preceding discussion showed that it is impossible to confirm, unambiguously, that all N particles have been prepared in the same state, with non-zero success probability whenever they are identical, unless the possible single-particle states are linearly independent. However, there is the possibility that this restriction does not apply to unambiguous confirmation of when this is not the case, i.e. of when the states of at least two particles are different. In this section, we shall describe a measurement that enables us to achieve this. For all N-particle states where the individual particles are not all in the same state, our measurement unambiguously confirms this fact with non-zero probability. The fact that it does this for all such states allows us to call it a universal measurement. We will describe its key properties and then prove that under a large set of circumstances, it is actually optimal. This is to say that it gives the minimum average probability of giving an inconclusive result. One particularly important set of circumstances under which it is optimal is when the possible states of each of the N systems are uniformly distributed, that is, they are completely arbitrary and unknown.

Without loss of generality, we may restrict our measurement to have only two possible outcomes, 'different' and '?', having POVM elements E_{diff} and $E_?$, respectively. Any POVM element E_{same} indicating that all particles are in the same state can simply be added to $E_?$ resulting in a coarse graining of these two outcomes. This will also apply to the analysis in section 5.

Let the N particles be prepared in arbitrary pure states in $\mathcal{H}$. We shall denote the state of particle j by $|\psi_j\rangle$. The state vector of the entire N-particle system can then be any product state in $\mathcal{H}_{\text{tot}}$. We write it as

$$|\Psi\rangle = \bigotimes_{j=1}^{N} |\psi_j\rangle. \tag{4.1}$$

From equation (2.2), we see that, for the N-particle state $|\Psi\rangle$, the probability of obtaining conclusive difference detection is

$$P_{\text{diff}}(\Psi) = \langle\Psi|E_{\text{diff}}|\Psi\rangle. \tag{4.2}$$

For any N-particle state $|\Psi\rangle$ which is of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$ for some single-particle state $|\psi\rangle \in \mathcal{H}$, we see that all N particles are in the same state. If this state has non-zero probability of being prepared, then the condition of unambiguity implies that $P_{\text{diff}}(\Psi) = 0$ for such a state. However, we require that $P_{\text{diff}}(\Psi) > 0$ for every N-particle product state $|\Psi\rangle$ which is not of this form.

To describe our measurement which accomplishes this task, it will be necessary to decompose the total Hilbert space $\mathcal{H}_{\rm tot}$ into two subspaces. These are the symmetric subspace and its orthogonal complement, which we shall refer to as the asymmetric subspace. We denote these by $\mathcal{H}_{\rm sym}$ and $\mathcal{H}_{\rm asym}$, respectively. Let $P(\mathcal{H}_{\rm sym})$ and $P(\mathcal{H}_{\rm asym})$ be the projectors onto $\mathcal{H}_{\rm sym}$ and $\mathcal{H}_{\rm asym}$, respectively. The POVM elements corresponding to conclusive detection of at least one difference and the inconclusive result are

$$E_{\rm diff} = P(\mathcal{H}_{\rm asym}), \tag{4.3}$$

$$E_? = P(\mathcal{H}_{\rm sym}). \tag{4.4}$$

We will now proceed to show that this measurement has the properties described above.

There are two things that we must prove. Firstly, we must prove that if all N particles are in the same state, i.e. if $|\Psi\rangle = |\psi\rangle^{\otimes N}$ for some $|\psi\rangle \in \mathcal{H}$, then the probability of obtaining a conclusive difference detection is zero. Secondly, we must show that for every product state $|\Psi\rangle$ which is not of this form, the probability of obtaining a conclusive result is non-zero. Equivalently, the probability of obtaining an inconclusive result is unity if and only if all particles are in the same state. We shall prove that our measurement satisfies these requirements.

To do so, we will obtain an explicit expression for the probability of obtaining an inconclusive result, and show that this can only take the value of unity when all N particles have been prepared in the same state. This expression will also be investigated in greater detail in section 6. To this end, let us denote by $S(N)$ the symmetric (or permutation) group of degree N. The symmetric tensor product of $|\psi_1\rangle, \ldots, |\psi_N\rangle$ is defined as [15]

$$|\psi_1\rangle \vee \cdots \vee |\psi_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S(N)} |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle. \tag{4.5}$$

For the state $|\Psi\rangle$ in equation (4.1), the projector onto the symmetric subspace acts as follows [15]:

$$\begin{aligned} P(\mathcal{H}_{\rm sym})|\Psi\rangle &= \frac{1}{\sqrt{N!}} |\psi_1\rangle \vee \cdots \vee |\psi_N\rangle \\ &= \frac{1}{N!} \sum_{\sigma \in S(N)} |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle. \end{aligned} \tag{4.6}$$

For the inconclusive result POVM element in equation (4.4), the probability that this result is obtained for the state $|\Psi\rangle$ in equation (4.1) is

$$\begin{aligned} P_?(\Psi) &= \langle \Psi | P(\mathcal{H}_{\rm sym}) | \Psi \rangle \\ &= \frac{1}{N!} \langle \psi_1 | \otimes \cdots \otimes \langle \psi_N | \sum_{\sigma \in S(N)} |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle \\ &= \frac{1}{N!} \sum_{\sigma \in S(N)} \langle \psi_1 | \psi_{\sigma(1)} \rangle \cdots \langle \psi_N | \psi_{\sigma(N)} \rangle. \end{aligned} \tag{4.7}$$

Consider now the Gram matrix $\Gamma = (\gamma_{ij})$ where $\gamma_{ij} = \langle \psi_i | \psi_j \rangle$. From equation (4.7) it follows that

$$P_?(\Psi) = \frac{1}{N!} \sum_{\sigma \in S(N)} \gamma_{1\sigma(1)} \ldots \gamma_{N\sigma(N)}. \tag{4.8}$$

If we now make use of the following definition of the permanent of an $N \times N$ matrix $A = (a_{ij})$,

$$\mathrm{per}(A) = \sum_{\sigma \in S(N)} a_{1\sigma(1)} \cdots a_{N\sigma(N)}, \tag{4.9}$$

then we finally obtain

$$P_?(\Psi) = \frac{1}{N!}\,\mathrm{per}(\Gamma). \tag{4.10}$$

For the probability of obtaining an inconclusive result to be unity, it is clear from equation (4.10) that we must have $\mathrm{per}(\Gamma) = N!$. Consider now the summation in equation (4.8), which is the Gram matrix permanent. It is a sum of $N!$ terms (since there are $N!$ possible permutations of N objects), each of which is bounded in absolute value by 1 (since they are all products of inner products of normalized states). $\mathrm{Per}(\Gamma)$ attains the value of $N!$ if and only if each of the terms in this summation is equal to 1. For this to be true, it is necessary that $|\psi_j\rangle = \mathrm{e}^{\mathrm{i}\phi_j}|\psi\rangle$, for some $|\psi\rangle \in \mathcal{H}$ and some phases ϕ_j, i.e. the states of the N particles are identical. To prove that this is also sufficient, we note that $\mathrm{per}(\Gamma) = N!$ if $\phi_j = 0$. However, these phase factors do not influence the value of $\mathrm{per}(\Gamma)$. The reason is that they accumulate to form an overall phase of $|\Psi\rangle$, which disappears in the evaluation of the expectation value of $P(\mathcal{H}_{\mathrm{sym}})$. Consequently, whenever all N particles are in the same state, regardless of possible phase factors, we have $\mathrm{per}(\Gamma) = N!$. This proves that the probability of obtaining an inconclusive result is unity if and only if all particles are in the same state.

4.2. Proof of optimality

We will now show that, under a large set of circumstances, the measurement described above is optimal. By optimal, we mean that it attains the minimum average probability of obtaining an inconclusive result. For any particular measurement strategy, to determine the average probability of an inconclusive result, it is necessary to know the *a priori* probabilities of the states. The most general situation that can be considered is where the set of possible states is the set of all product states in the space $\mathcal{H}_{\mathrm{tot}}$, that is, any state of the form $|\Psi\rangle$ shown in equation (4.1). The prior distribution of these states is described by an *a priori* probability density (henceforth simply density function) which we denote by $q(\psi_1, \ldots, \psi_N)$. This function is non-negative and normalized as

$$\int \mathrm{d}\sigma(\psi_1)\ldots\mathrm{d}\sigma(\psi_N)q(\psi_1, \ldots, \psi_N) = 1 \tag{4.11}$$

where each integration is with respect to the single-particle, unitarily invariant Haar measure $\mathrm{d}\sigma$. This measure is normalized, i.e. $\int \mathrm{d}\sigma(\psi) = 1$. The total probability of obtaining an inconclusive result is simply the average of the inconclusive result probability over the set of N-particle product states $|\Psi\rangle$ with respect to the density function q, that is,

$$\begin{aligned}\overline{P_?(\Psi)} &= \int \mathrm{d}\sigma(\psi_1)\ldots\mathrm{d}\sigma(\psi_N)q(\psi_1, \ldots, \psi_N)P_?(\Psi)\\ &= \mathrm{Tr}(\varrho E_?)\end{aligned} \tag{4.12}$$

where we have defined the N-particle density operator

$$\varrho = \int \mathrm{d}\sigma(\psi_1)\ldots\mathrm{d}\sigma(\psi_N)q(\psi_1, \ldots, \psi_N)\left(\bigotimes_{j=1}^{N}|\psi_j\rangle\langle\psi_j|\right). \tag{4.13}$$

We will now consider the special case where $q(\psi, \ldots, \psi) > 0$ for all $|\psi\rangle \in \mathcal{H}$. That is, all product states of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$, where all N particles are in the same state, are possible. We will show that when this condition is satisfied, the measurement described by the POVM elements in equations (4.3) and (4.4) attains the minimum average probability of inconclusive results. To show this, we will first prove that if $q(\psi, \ldots, \psi) > 0\ \forall|\psi\rangle \in \mathcal{H}$, then the condition of unambiguity is equivalent to the support of the POVM element E_{diff} being

a subspace of $\mathcal{H}_{\rm asym}$.[5] We will then show that, under conditions of optimality, $E_{\rm diff}$ may be taken to be the projector onto $\mathcal{H}_{\rm asym}$.

To prove the first of these assertions, if $q(\psi, \ldots, \psi) > 0 \; \forall |\psi\rangle \in \mathcal{H}$, then any state of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$ can occur. If the result of the measurement is to be unambiguous, then we must ensure that none of these states give rise to a conclusive result indicating difference. Formally, this requirement can be expressed as

$$(\langle\psi|^{\otimes N}) E_{\rm diff} (|\psi\rangle^{\otimes N}) = 0 \tag{4.14}$$

for all $|\psi\rangle \in \mathcal{H}$. We now make use of the fact that the projector onto the symmetric subspace may be written as

$$P(\mathcal{H}_{\rm sym}) = D(\mathcal{H}_{\rm sym}) \int {\rm d}\sigma(\psi) (|\psi\rangle\langle\psi|)^{\otimes N}. \tag{4.15}$$

Here, $D(\mathcal{H}_{\rm sym})$ is the dimensionality of $\mathcal{H}_{\rm sym}$. Its value is

$$D(\mathcal{H}_{\rm sym}) = \binom{D(\mathcal{H}) + N - 1}{N}. \tag{4.16}$$

We now make use of equations (4.14) and (4.15) and calculate ${\rm Tr}[P(\mathcal{H}_{\rm sym}) E_{\rm diff}]$, finding that

$${\rm Tr}[P(\mathcal{H}_{\rm sym}) E_{\rm diff}] = 0. \tag{4.17}$$

The trace of the product of two positive operators, such as we have here, is zero if and only if each eigenvector of one of the operators whose corresponding eigenvalue is non-zero is orthogonal to all eigenvectors of the other operator with non-zero eigenvalues. This implies that the eigenvectors of $E_{\rm diff}$ with non-zero eigenvalues must be orthogonal to $\mathcal{H}_{\rm sym}$. So, these eigenvectors must lie in the orthogonal complement of $\mathcal{H}_{\rm sym}$, which is $\mathcal{H}_{\rm asym}$. This proves our first assertion.

To prove the second, we make use of the fact that, as a consequence of the above argument, $E_{\rm diff}$ can be spectrally decomposed in terms of an orthonormal basis for $\mathcal{H}_{\rm asym}$,

$$E_{\rm diff} = \sum_{r=1}^{D(\mathcal{H}_{\rm asym})} e_r |e_r\rangle\langle e_r| \tag{4.18}$$

where $0 \leqslant e_r \leqslant 1$ and the $|e_r\rangle$ form an orthonormal basis for $\mathcal{H}_{\rm asym}$. Also, $D(\mathcal{H}_{\rm asym})$ is the dimensionality of $\mathcal{H}_{\rm asym}$, which is easily seen to be equal to $D(\mathcal{H})^N - D(\mathcal{H}_{\rm sym})$. If we now substitute this expression into equation (4.12) and make use of the resolution of the identity in equation (2.3), we find that

$$\begin{aligned} \overline{P_?(\Psi)} &= {\rm Tr}(\varrho E_?) = 1 - {\rm Tr}(\varrho E_{\rm diff}) \\ &= 1 - \sum_{r=1}^{D(\mathcal{H}_{\rm asym})} e_r \langle e_r|\varrho|e_r\rangle \\ &\geqslant 1 - \sum_{r=1}^{D(\mathcal{H}_{\rm asym})} \langle e_r|\varrho|e_r\rangle \\ &= 1 - {\rm Tr}[\varrho P(\mathcal{H}_{\rm asym})]. \end{aligned} \tag{4.19}$$

That is, the minimum value of $\overline{P_?(\Psi)}$ is obtained when $E_{\rm diff}$ is the projector onto $\mathcal{H}_{\rm asym}$. From the resolution of the identity in equation (2.3), it follows that the corresponding inconclusive result POVM element $E_?$ must be equal to $P(\mathcal{H}_{\rm sym})$. So, we have shown that, under the

[5] We take the support of a positive operator to be the subspace spanned by its eigenstates corresponding to non-zero eigenvalues.

specified conditions, the measurement described by the POVM elements in equations (4.3) and (4.4) is optimal. Equation (4.19) can be written in the form

$$\overline{P_?(\Psi)}(\min) = \mathrm{Tr}[\varrho P(\mathcal{H}_{\mathrm{sym}})]. \tag{4.20}$$

This expression can be simplified if the density function factorizes, i.e. if it is of the form

$$q(\psi_1, \ldots, \psi_N) = \prod_{j=1}^{N} q_j(\psi_j). \tag{4.21}$$

When this is the case, we can define the single-particle density operators

$$\rho_j = \int \mathrm{d}\sigma(\psi) q_j(\psi) |\psi\rangle\langle\psi| \tag{4.22}$$

so that $\varrho = \otimes_{j=1}^{N} \rho_j$. This enables us to write the minimum average inconclusive result probability as

$$\overline{P_?(\Psi)}(\min) = \mathrm{Tr}\left[\left(\otimes_{j=1}^{N} \rho_j\right) P(\mathcal{H}_{\mathrm{sym}})\right]. \tag{4.23}$$

Let us now simplify matters further by assuming that the single-particle density functions $q_j(\psi)$ are all equal to some density function $q(\psi)$. When this is the case, the single-particle density operators ρ_j are all equal to some density operator $\rho = \int \mathrm{d}\sigma(\psi) q(\psi) |\psi\rangle\langle\psi|$. Under such circumstances it can be shown that $\overline{P_?(\Psi)}(\min)$ is the Nth complete symmetric polynomial in the eigenvalues of ρ [15]. This takes an even simpler form when ρ is the maximally mixed state $1_{\mathcal{H}}/D(\mathcal{H})$, where $1_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. This situation arises when the product states $|\Psi\rangle$ are uniformly distributed, and so the states of the N particles are arbitrary and equally probable. When this is the case, we obtain

$$\overline{P_?(\Psi)}(\min) = \frac{1}{D(\mathcal{H})^N} \mathrm{Tr}[P(\mathcal{H}_{\mathrm{sym}})] = \frac{D(\mathcal{H}_{\mathrm{sym}})}{D(\mathcal{H})^N}. \tag{4.24}$$

5. Unambiguous confirmation of all particles having different states

5.1. Requirement of linear independence

In the preceding section, we addressed the problem of unambiguous detection of at least one difference among the states of N particles. An equally important issue to consider is the unambiguous determination of whether or not the states of all N particles are different. It is to this matter that we turn our attentions here.

The situation we will consider here is as follows. Each of the N particles is prepared in an element of the set of M possible pure states $\{|\psi_\mu\rangle\}$, like we had in section 3. Denoting the state of particle j by $\left|\psi_{\mu_j}\right\rangle$, it follows that the state of the N-particle system is $\left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle$ given by equation (3.1). All these states are taken to occur with non-zero probability. Our aim is to devise a measurement which enables us to unambiguously determine when all N particles have been prepared in different states. It must give a conclusive result with non-zero probability for the state $\left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle$ if and only if the μ_j all have different values.

Prior to discussing this matter in detail, we should point out that we require $M \geqslant N$. Were this not the case, then the number of particles would exceed the number of possible single-particle states and so, unavoidably, at least two particles would always be in the same state and so the states of the N particles could not all be different. It is worth making the observation that this is a consequence of the well-known 'pigeonhole principle', with particles playing the role of pigeons and states playing that of holes.

As was the case in the preceding section, the measurement we require to perform this task will have two possible outcomes, each with a corresponding POVM element. It must have a POVM element E_{diff} corresponding to unambiguous confirmation that the N particles are all in different states, and a POVM element $E_?$ giving inconclusive results. These two operators must be positive and satisfy the resolution of the identity in equation (2.3). The probabilities of obtaining these two outcomes for a particular N-particle state $\left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle$ are

$$P_{\mathrm{diff}}\left(\Psi_{\mu_1,\ldots,\mu_N}\right) = \left\langle\Psi_{\mu_1,\ldots,\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle, \tag{5.1}$$

$$P_?\left(\Psi_{\mu_1,\ldots,\mu_N}\right) = \left\langle\Psi_{\mu_1,\ldots,\mu_N}\right| E_? \left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle. \tag{5.2}$$

The result of our measurement must be unambiguous. This means that a conclusive result can only be obtained with non-zero probability if the $\left|\psi_{\mu_r}\right\rangle$ or, equivalently, the μ_r, are all different. We then require

$$\left\langle\Psi_{\mu_1,\ldots,\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle = 0 \tag{5.3}$$

if any of the μ_j are equal. We also require that if $\left|\psi_{\mu_1}\right\rangle, \ldots, \left|\psi_{\mu_N}\right\rangle$ are all different then $P_{\mathrm{diff}}\left(\Psi_{\mu_1,\ldots,\mu_N}\right) > 0$.

Here, we will show that, for a measurement satisfying the above conditions to be possible, it is necessary that no N-element subset of the set of M possible single-particle states $\{|\psi_\mu\rangle\}$ is linearly dependent. Except when $M = N$, this condition is weaker than the linear independence of all M possible single-particle states, which was our necessary and sufficient condition for unambiguous detection of at least one difference in section 3. We will subsequently show that the linear independence of each N-element subset is also a sufficient condition. In fact, we will show that there exists a single, universal measurement which can be used to confirm, with non-zero probability, that all N particles have been prepared in different states whenever this linear independence condition is satisfied.

To prove our first assertion, let us assume that N of these states, $\left|\psi_{\mu_1}\right\rangle, \ldots, \left|\psi_{\mu_N}\right\rangle$, are all different, but linearly dependent. From these states, we construct the N-particle state $\left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle$ according to equation (3.1). The linear dependence of the single-particle states implies that the state of one of the particles may be expressed as a linear combination of the states of the other $N-1$ particles. We can label the N particles in such a way that this particle is particle 1. We may then write

$$\left|\psi_{\mu_1}\right\rangle = \sum_{r=2}^{N} f_r \left|\psi_{\mu_r}\right\rangle \tag{5.4}$$

for some complex coefficients f_r. If we substitute this expression for $\left|\psi_{\mu_1}\right\rangle$ into equation (5.1), we obtain

$$P_{\mathrm{diff}}\left(\Psi_{\mu_1,\ldots,\mu_N}\right) = \sum_{r,r'=2}^{N} f_{r'}^* f_r \left\langle\Psi_{\mu_{r'}\mu_2,\ldots,\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_r\mu_2\ldots\mu_N}\right\rangle. \tag{5.5}$$

We will now see that the value of this expression must be zero, by proving that the inner products in the summation vanish. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\left|\left\langle\Psi_{\mu_{r'}\mu_2\ldots\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_r\mu_2\ldots\mu_N}\right\rangle\right|^2 &\leqslant \left\langle\Psi_{\mu_{r'}\mu_2\ldots\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_{r'}\mu_2\ldots\mu_N}\right\rangle \left\langle\Psi_{\mu_r\mu_2\ldots\mu_N}\right| E_{\mathrm{diff}} \left|\Psi_{\mu_r\mu_2\ldots\mu_N}\right\rangle \\ &= P_{\mathrm{diff}}\left(\Psi_{\mu_{r'}\mu_2\ldots\mu_N}\right) P_{\mathrm{diff}}\left(\Psi_{\mu_r\mu_2\ldots\mu_N}\right).\end{aligned} \tag{5.6}$$

However, both factors in the last line of this expression must be zero. This is because they are both probabilities of obtaining a conclusive confirmation of all particles being in different states, but where the state of the first particle is identical to the state of one of the other $N-1$ particles. In the first probability, it is particle r' while in the second, it is particle r. It follows

from the requirement of unambiguity that these vanish, and so, by the Cauchy–Schwarz inequality, do the inner products in the sum in equation (5.5), and therefore the sum itself. So, we have shown that to confirm unambiguously that all N particles have been prepared in different states, it is necessary for the states of the N particles to form a linearly independent set. Since this must be true for every N-particle initial state $\left|\Psi_{\mu_1,\ldots,\mu_N}\right\rangle$, we see that this is equivalent to the requirement that no N-element subset of the set of M possible single-particle states $\{|\psi_\mu\rangle\}$ is linearly dependent.

It will be shown below that this condition is also sufficient for unambiguous confirmation that all N particles are in different states. We stated above that this condition is, except when $M = N$, weaker than the requirement of the set of all M single-particle states to be linearly independent. A simple example of a set of single-particle states which satisfies the first of these linear independence conditions but not the second is as follows. Suppose that $M = N + 1$ and that the first N of the states $|\psi_\mu\rangle$ form an orthonormal basis for $\mathcal{H}$, but where $|\psi_M\rangle = \sum_{\mu=1}^{N} a_\mu |\psi_\mu\rangle$, where the a_μ are all non-zero. Clearly, the set of all M single-particle states $|\psi_\mu\rangle$ is linearly dependent. However, this set has no N-element linearly dependent subset. So, when the individual particles are prepared in states in this set according to the conditions described here and in section 3, it is impossible to unambiguously confirm that they have all been prepared in the same state, although it is possible to unambiguously confirm that their states are all different.

5.2. Universal measurement strategy

We will now show that this condition of linear independence is also sufficient. To do so, we will construct a measurement which, with non-zero probability, unambiguously confirms that all N particles have been prepared in different states whenever the above linear independence condition is satisfied. For the sake of maximum generality, we no longer restrict ourselves to the case of each particle being prepared in one of a finite number of possible states. Instead, we allow it to be any pure state in $\mathcal{H}$. So, employing again the notation of section 4, let us denote the initial state of particle j by $|\psi_j\rangle$ and the entire N-particle state by $|\Psi\rangle$ given by equation (4.1). In the measurement we shall consider, it will be necessary to decompose the total Hilbert space $\mathcal{H}_{\mathrm{tot}}$ into two subspaces, the antisymmetric subspace, which we shall denote by $\mathcal{H}_{\mathrm{anti}}$, and its orthogonal complement, the non-antisymmetric subspace, which we shall denote by $\mathcal{H}_{\mathrm{na}}$. Let $P(\mathcal{H}_{\mathrm{anti}})$ and $P(\mathcal{H}_{\mathrm{na}})$ be the projectors onto $\mathcal{H}_{\mathrm{anti}}$ and $\mathcal{H}_{\mathrm{na}}$, respectively. Consider the POVM with the following elements:

$$E_{\mathrm{diff}} = P(\mathcal{H}_{\mathrm{anti}}), \tag{5.7}$$

$$E_{?} = P(\mathcal{H}_{\mathrm{na}}). \tag{5.8}$$

We will now show that this measurement satisfies our requirements. To do so, we will derive an explicit expression for the difference detection probability, which will be discussed in more detail in section 6. Consider again $S(N)$, the symmetric group of degree N. The antisymmetric tensor product of $|\psi_1\rangle, \ldots, |\psi_N\rangle$ is defined as [15]

$$|\psi_1\rangle \wedge \cdots \wedge |\psi_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S(N)} \varepsilon_\sigma |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle \tag{5.9}$$

where ε_σ is equal to $+1$ or -1 when σ is an even or odd permutation, respectively. For the state $|\Psi\rangle$ given by equation (4.1), the projector onto the antisymmetric subspace acts as follows:

$$P(\mathcal{H}_{\text{anti}})|\Psi\rangle = \frac{1}{\sqrt{N!}}|\psi_1\rangle \wedge \cdots \wedge |\psi_N\rangle$$
$$= \frac{1}{N!}\sum_{\sigma\in S(N)} \varepsilon_\sigma |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle. \tag{5.10}$$

From this and equations (2.2) and (5.8), we see that the probability of obtaining a conclusive difference detection for the N-particle state $|\Psi\rangle$ is given by

$$P_{\text{diff}}(\Psi) = \langle\Psi|P(\mathcal{H}_{\text{anti}})|\Psi\rangle$$
$$= \frac{1}{N!}\langle\psi_1| \otimes \cdots \otimes \langle\psi_N| \sum_{\sigma\in S(N)} \varepsilon_\sigma |\psi_{\sigma(1)}\rangle \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle$$
$$= \frac{1}{N!}\sum_{\sigma\in S(N)} \varepsilon_\sigma \langle\psi_1|\psi_{\sigma(1)}\rangle \cdots \langle\psi_N|\psi_{\sigma(N)}\rangle. \tag{5.11}$$

Let us now recall the Gram matrix $\Gamma = (\gamma_{ij})$ where $\gamma_{ij} = \langle\psi_i|\psi_j\rangle$. It follows from equation (5.11) that

$$P_{\text{diff}}(\Psi) = \frac{1}{N!}\sum_{\sigma\in S(N)} \varepsilon_\sigma \gamma_{1\sigma(1)} \cdots \gamma_{N\sigma(N)}. \tag{5.12}$$

If we now make use of the following definition of the determinant of an $N \times N$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{\sigma\in S(N)} \varepsilon_\sigma a_{1\sigma(1)} \cdots a_{N\sigma(N)}, \tag{5.13}$$

we finally obtain

$$P_{\text{diff}}(\Psi) = \frac{1}{N!}\det(\Gamma). \tag{5.14}$$

So, for this measurement, the probability of obtaining a conclusive result is proportional to the determinant of the Gram matrix of the states of the N particles.

Now, the determinant of a Gram matrix is non-zero if and only if the vectors forming it are linearly independent [16]. This proves the sufficiency of the linear independence of each N-element subset of the M possible single-particle states to imply that the conclusive outcome will have a non-zero probability. The fact that the Gram matrix determinant is zero when the states of the N particles are linearly dependent is also sufficient to guarantee that the measurement result is unambiguous. This is a simple consequence of the fact that, whenever two or more particles are in the same state, the states of the N particles are clearly linearly dependent.

5.3. Proof of optimality

We saw above that to confirm unambiguously that all N particles have been prepared in different states, a necessary and sufficient condition is that the states of the N particles are linearly independent. The sufficiency part of our proof used an explicit measurement strategy which can always achieve this confirmation with non-zero probability if this linear independence condition is satisfied. This measurement is a projective measurement which perfectly discriminates between the antisymmetric and non-antisymmetric subspaces of $\mathcal{H}_{\text{tot}}$.

We will now address the issue of the optimum measurement for detecting when all N particles are in different states. That is, we would like to determine the measurement which attains the minimum average probability of an inconclusive result. As was the case in section 4,

to calculate this probability, we require knowledge of the density function $q(\psi_1, \ldots, \psi_N)$. With this, we calculate the N-particle density operator ϱ using equation (4.13). We then obtain the average probability of an inconclusive result

$$\overline{P_?(\Psi)} = \mathrm{Tr}(\varrho E_?) \tag{5.15}$$

as before.

Here, we will consider the situation where the density function is everywhere non-zero, i.e., where $q(\psi_1, \ldots, \psi_N) > 0$ for all $|\psi_j\rangle \in \mathcal{H}$. When this is the case, all N-particle product states are possible. We will show that, under these circumstances, one measurement which attains the minimum value of $\overline{P_?(\Psi)}$ is the measurement described by the POVM elements in equations (5.7) and (5.8). We shall, in fact, restrict our attention to cases when $N \geqslant 3$. The reason for this is that, if $N = 2$, then unambiguous confirmation that both particles are in different states is equivalent to unambiguous confirmation that they are not both in the same state. This problem was solved in the preceding section, and it was found that the optimum measurement is a projective measurement with the POVM elements $E_? = P(\mathcal{H}_{\mathrm{sym}})$ and $E_{\mathrm{diff}} = P(\mathcal{H}_{\mathrm{asym}})$. However, for $N = 2$, we have $\mathcal{H}_{\mathrm{sym}} = \mathcal{H}_{\mathrm{na}}$ and $\mathcal{H}_{\mathrm{asym}} = \mathcal{H}_{\mathrm{anti}}$, so for the special case of $N = 2$ we have already proven the above claim (indeed under more restrictive conditions) in section 4.

In proving this claim for $N \geqslant 3$, we shall first prove that under the specified conditions, the requirement of unambiguity implies that the support of E_{diff} must be a subspace of $\mathcal{H}_{\mathrm{anti}}$. We will then show that, under conditions of optimality, E_{diff} may be taken to be the projector onto $\mathcal{H}_{\mathrm{anti}}$.

To prove the first of these points, let $|e\rangle$ be an eigenstate of E_{diff} with non-zero eigenvalue. For any N-particle product state $|\Psi\rangle$ where at least two of the $|\psi_j\rangle$ are identical, the positivity of E_{diff} and the requirement of unambiguity imply that we must have

$$\langle\Psi|e\rangle = 0. \tag{5.16}$$

To proceed, let us partition the set of N particles into two subsets, α and $\bar{\alpha}$. The former consists of any pair of particles and the latter consists of the remaining $N - 2$. Let us denote the Hilbert spaces of these multiparticle systems by $\mathcal{H}_\alpha$ and $\mathcal{H}_{\bar{\alpha}}$, respectively. We can express any N-particle pure state in Schmidt decomposition form, where each element of the Schmidt basis is a tensor product of one state from each of the spaces $\mathcal{H}_\alpha$ and $\mathcal{H}_{\bar{\alpha}}$. Applying this idea to the state $|e\rangle$, we see that we may write

$$|e\rangle = \sum_{r=1}^{D^2} c_r |x_r\rangle \otimes |y_r\rangle. \tag{5.17}$$

Here, the set $\{|x_r\rangle\}$ is an orthonormal basis for $\mathcal{H}_\alpha$ and the $|y_r\rangle$ are D^2 orthonormal states in $\mathcal{H}_{\bar{\alpha}}$. Now suppose that the states of the two particles in α are identical. If this is so, then we can write $|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes |\Phi\rangle$ for some $|\psi\rangle \otimes |\psi\rangle \in \mathcal{H}_\alpha$ and some $(N-2)$-particle product state $|\Phi\rangle \in \mathcal{H}_{\bar{\alpha}}$. Substituting this into equation (5.16) and making use of equation (5.17) we obtain

$$\sum_{r=1}^{D^2} c_r (\langle\psi| \otimes \langle\psi|)|x_r\rangle\langle\Phi|y_r\rangle = 0. \tag{5.18}$$

This must be true for all $(N-2)$-particle product states $|\Phi\rangle \in \mathcal{H}_{\bar{\alpha}}$ and for all $|\psi\rangle \in \mathcal{H}$. From equation (5.18), we can deduce that for any r such that $c_r \neq 0$, we have $(\langle\psi| \otimes \langle\psi|)|x_r\rangle = 0$ for all $|\psi\rangle \in \mathcal{H}$. It is quite simple to prove this. Consider the vector $\sum_{r'} c_{r'}[(\langle\psi| \otimes \langle\psi|)|x_{r'}\rangle]|y_{r'}\rangle \in \mathcal{H}_{\bar{\alpha}}$. From equation (5.18), we see that this vector is orthogonal to all product states in $\mathcal{H}_{\bar{\alpha}}$.

Since any vector in $\mathcal{H}_{\bar{\alpha}}$ can be expressed as a linear combination of product states in $\mathcal{H}_{\bar{\alpha}}$, it must therefore be orthogonal to all vectors in $\mathcal{H}_{\bar{\alpha}}$. This can only be true if it is the zero vector

$$\sum_{r'=1}^{D^2} c_{r'}[(\langle\psi| \otimes \langle\psi|)|x_{r'}\rangle]|y_{r'}\rangle = 0. \tag{5.19}$$

If we now take the inner product throughout this expression with any $|y_r\rangle$ for which $c_r \neq 0$, we finally obtain $(\langle\psi| \otimes \langle\psi|)|x_r\rangle = 0$ for all $|\psi\rangle \in \mathcal{H}$, as we claimed.

From equation (4.15), we see that the projector onto the symmetric subspace of $\mathcal{H}_\alpha$ is $\binom{D(\mathcal{H})+1}{2} \int \mathrm{d}\sigma(\psi)(|\psi\rangle\langle\psi|)^{\otimes 2}$. Combining this expression with $(\langle\psi| \otimes \langle\psi|)|x_r\rangle = 0$, it follows that if $c_r \neq 0$ then $|x_r\rangle$ is orthogonal to the symmetric subspace of $\mathcal{H}_\alpha$. It must therefore be an element of its orthogonal complement. For a bipartite quantum system such as α, the orthogonal complement of the symmetric subspace is the antisymmetric subspace. This implies that the state $|e\rangle$ is antisymmetric under exchange of the two particles in α. However, the set α comprises an arbitrary pair chosen from all N particles, and so the state $|e\rangle$ must be antisymmetric under exchange of any pair of particles. This implies that it must lie in the N-particle antisymmetric subspace $\mathcal{H}_{\mathrm{anti}}$. Finally, since $|e\rangle$ is an arbitrary eigenstate of E_{diff} with non-zero eigenvalue, it follows that all eigenstates of this operator with non-zero eigenvalue lie in $\mathcal{H}_{\mathrm{anti}}$. In other words, the support of E_{diff} is a subspace of $\mathcal{H}_{\mathrm{anti}}$.

To prove the second assertion, we make use of the fact that, in view of the above argument, E_{diff} can be spectrally decomposed in terms of an orthonormal basis for $\mathcal{H}_{\mathrm{anti}}$:

$$E_{\mathrm{diff}} = \sum_{r=1}^{D(\mathcal{H}_{\mathrm{anti}})} e_r |e_r\rangle\langle e_r|. \tag{5.20}$$

Here, $0 \leqslant e_r \leqslant 1$ and the $|e_r\rangle$ form an orthonormal basis for $\mathcal{H}_{\mathrm{anti}}$. Also, $D(\mathcal{H}_{\mathrm{anti}})$ is the dimensionality of $\mathcal{H}_{\mathrm{anti}}$ and it has the value

$$D(\mathcal{H}_{\mathrm{anti}}) = \binom{D(\mathcal{H})}{N}. \tag{5.21}$$

Substitution of the expression in equation (5.20) for E_{diff} into equation (5.15) and making use of the resolution of the identity in equation (2.3) gives

$$\begin{aligned}
\overline{P_?(\Psi)} &= \mathrm{Tr}(\varrho E_?) = 1 - \mathrm{Tr}(\varrho E_{\mathrm{diff}}) \\
&= 1 - \sum_{r=1}^{D(\mathcal{H}_{\mathrm{anti}})} e_r \langle e_r|\varrho|e_r\rangle \\
&\geqslant 1 - \sum_{r=1}^{D(\mathcal{H}_{\mathrm{anti}})} \langle e_r|\varrho|e_r\rangle \\
&= 1 - \mathrm{Tr}[\varrho P(\mathcal{H}_{\mathrm{anti}})].
\end{aligned} \tag{5.22}$$

So, the minimum value of $\overline{P_?(\Psi)}$ occurs when the difference detection POVM element E_{diff} is the projector onto the antisymmetric subspace $\mathcal{H}_{\mathrm{anti}}$. From the resolution of the identity in equation (2.3), we see that the corresponding POVM element for inconclusive results, $E_?$, must be equal to $P(\mathcal{H}_{\mathrm{na}})$, the projector onto the non-antisymmetric subspace $\mathcal{H}_{\mathrm{na}}$. We have shown that, under the specified conditions, the measurement described by the POVM elements in equations (5.7) and (5.8) is optimal and that

$$\overline{P_?(\Psi)}(\mathrm{min}) = \mathrm{Tr}[\varrho P(\mathcal{H}_{\mathrm{na}})]. \tag{5.23}$$

As was the case for the measurement described in the preceding section for universally detecting a single difference, this probability can be simplified if the density function factorizes

according to equation (4.21). When this is so, the N-particle density operator takes the form $\varrho = \otimes_{j=1}^{N} \rho_j$ where the single-particle density operators ρ_j are given by equation (4.22). This enables us to write the minimum average inconclusive result probability as

$$\overline{P_?(\Psi)}(\min) = 1 - \mathrm{Tr}\left[\left(\otimes_{j=1}^{N} \rho_j\right) P(\mathcal{H}_{\mathrm{anti}})\right]. \tag{5.24}$$

In particular, if the ρ_j are all equal to some single-particle density operator ρ, then the last term in this above expression has the form $\mathrm{Tr}[\rho^{\otimes N} P(\mathcal{H}_{\mathrm{anti}})]$. This can be shown to be equal to the Nth elementary symmetric polynomial in the eigenvalues of ρ [15]. Finally, if ρ is the maximally mixed state $1_{\mathcal{H}}/D(\mathcal{H})$, then we obtain

$$\overline{P_?(\Psi)}(\min) = 1 - \frac{1}{D(\mathcal{H})^N} \mathrm{Tr}[P(\mathcal{H}_{\mathrm{anti}})] = 1 - \frac{D(\mathcal{H}_{\mathrm{anti}})}{D(\mathcal{H})^N}. \tag{5.25}$$

This situation arises when the N-particle product states $|\Psi\rangle$ are uniformly distributed, in which case the states of the N particles are arbitrary and equally probable.

6. Optimal universal difference detection and distinguishability

In sections 4 and 5, we considered optimal, universal difference detection. The aim in both sections was to determine measurement strategies which optimally detect differences among the states of N particles. For a broad range of circumstances, including when all N-particle product states are possible, we obtained the general form of the optimal strategies for detecting when at least two, and when all N particles are in different states. We also obtained expressions for the associated difference detection probabilities. For the measurement used to unambiguously determinine when the states of all N particles are different, the success probability for a particular N-particle product state $|\Psi\rangle = \otimes_{j=1}^{N} |\psi_j\rangle$ is given by equation (5.14), which we repeat here for convenience

$$P_{\mathrm{diff}}(\Psi) = \frac{1}{N!} \det(\Gamma). \tag{6.1}$$

For the measurement which detects when at least two particles are in different states, the inconclusive result probability for a particular N-particle product state is given by equation (4.10). From this, we see that the success probability is

$$P_{\mathrm{diff}}(\Psi) = 1 - \frac{1}{N!} \mathrm{per}(\Gamma). \tag{6.2}$$

For these measurements, the true figures of merit are the averages of these quantities over all product states with respect to the density function q. These probabilities are equal to 1 minus the corresponding minimum average inconclusive result probabilities, given by equations (5.25) and (4.24), respectively. Nevertheless, the success probabilities in equations (6.1) and (6.2) for a particular N-particle state do depend on the states of the individual particles through their Gram matrix Γ. In this section, we shall examine these success probabilities in more detail. Our focus will be on their state-dependent nature. It is natural to enquire as to what property of the states of the individual particles these probabilities relate to. Here, we shall investigate the possibility that, for a fixed N-particle product state $|\Psi\rangle$, these success probabilities can be viewed as measures of the distinguishability of the states of the individual particles $|\psi_j\rangle$, considered as possible states of a single particle. Our intuition is that, the higher the probability of obtaining a conclusive result, the more distinguishable these states are. Often, in discussions of distinguishability, the *a priori* probabilities of the states, as well as the states themselves, are taken into account. The reason is that distinguishability measures are typically probabilities, which we need to know the *a priori* probabilities of the possible states to evaluate. Here, we simply take all states to be equally probable.

To explore the validity of this idea, it is necessary to determine whether or not the probabilities in equations (6.1) and (6.2) have the properties which any distinguishability measure must have. As yet, no general set of necessary and sufficient criteria for a given quantity to be a suitable distinguishability measure have, to our knowledge, been proposed. However, certain intuitive necessary conditions are widely accepted. One is the fact that only orthogonal sets of states can be considered maximally distinguishable. Another is the fact that a distinguishability measure must be non-increasing under a deterministic quantum operation. We show that both of these success probabilities satisfy the first of these criteria. We also find that the second criterion is always satisfied by the probability of finding all states being different in equation (6.1). Whether or not this is also true in general for the success probability of finding at least one difference among the states in equation (6.2) is, we find, equivalent to an open conjecture in linear algebra.

We will first prove that the success probability in equation (6.1) for finding all states being different satisfies the above criteria. We will then carry out a corresponding analysis for the probability of confirming at least one difference in equation (6.2). Beginning with $P_{\mathrm{diff}}(\Psi)$ in equation (6.1), we are interested in whether or not this attains its maximum value when the states in the Gram matrix Γ are orthogonal. This is clearly equivalent to asking if the Gram matrix determinant maximizes for orthogonal states. This is indeed the case, and to prove it, we use Hadamard's inequality [22], which is satisfied by any positive $N \times N$ matrix $A = (a_{ij})$,

$$\det(A) \leqslant \prod_{i=1}^{N} a_{ii} \tag{6.3}$$

where the equality is satisfied if and only if A is diagonal.

All Gram matrices are positive. For normalized states, the diagonal elements of a Gram matrix are all equal to 1. It follows from the Hadamard inequality that $\det(\Gamma) \leqslant 1$, with the equality holding if and only if the off-diagonal elements of Γ vanish. This corresponds to the states $|\psi_j\rangle$ being orthogonal and therefore maximally distinguishable. So, $P_{\mathrm{diff}}(\Psi)$ in equation (6.1), considered as a distinguishability measure, correctly identifies the maximally distinguishable sets of states as being orthogonal sets.

We saw in section 5 that the determinant of Γ takes the value of zero if and only if the single-particle states are linearly dependent. For a positive matrix such as Γ, this is the minimum value of $\det(\Gamma)$. However, a linearly dependent set of states does not necessarily have zero distinguishability for all state discrimination strategies. For example, in quantum hypothesis testing, where we aim to discriminate among the states with minimum error probability, linearly dependent sets do not in general represent the worst case scenario. However, they do in other state discrimination strategies. In particular, in unambiguous state discrimination, the probability of discriminating among a linearly dependent set of states is zero. However, this is not particularly important. It is known that different distinguishability measures do not, in general, characterize the distinguishability of the same set of states in the same way. In particular, it is known that different distinguishability measures can impose different orderings on two sets of states. This matter is explored in more detail in [19].

What this implies is that, if we wish to further test the Gram matrix determinant for its viability as a distinguishability measure, then we should be judicious about which aspects of its behaviour we should assess. Quantum state discrimination has an operational aspect which is, to a certain extent, distinct from the actual act of distinguishing among quantum states. This is a constraint on the transformation of a distinguishability measure under a deterministic quantum operation. All distinguishability measures must be non-increasing under such operations. The reason for this is simple. If this condition were not satisfied, then we could increase the value of the distinguishability measure by incorporation of a deterministic quantum operation that

increases it into the state discrimination procedure. Deterministic quantum operations are represented mathematically by completely positive, linear, trace-preserving (CPLTP) maps. Quantities which have this non-increasing property are known as distinguishability monotones [20].

Clearly, it is of interest to determine whether or not the Gram matrix determinant is a distinguishability monotone. For this question to be meaningful, we must restrict our attention to situations where the initial pure states, which we shall write as $|\psi_j^1\rangle$, are transformed into another set of pure states, $|\psi_j^2\rangle$, since the Gram matrix is defined only for pure states.

It is known [17, 18, 20, 21] that a necessary and sufficient condition for the existence of a CPLTP map which deterministically transforms $|\psi_j^1\rangle$ into $|\psi_j^2\rangle$ for each $j = 1, \ldots, N$ is that there exists an $N \times N$ positive matrix Π such that

$$\Gamma_1 = \Gamma_2 \circ \Pi \tag{6.4}$$

where Γ_1 and Γ_2 are the Gram matrices of the initial and final sets of states respectively and '$\circ$' denotes the Hadamard product. The Hadamard product $A \circ B$ of two $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ has ij element $a_{ij}b_{ij}$, i.e., it is the entrywise product. We would like to know if equation (6.4) implies that

$$\det(\Gamma_1) \geqslant \det(\Gamma_2) \tag{6.5}$$

that is, whether or not the Gram matrix determinant is non-increasing under such a transformation. We shall now prove that this inequality is indeed always satisfied for such a transformation. To do so, we shall use Oppenheim's inequality [22, 23]. For any two positive $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, Oppenheim proved that

$$\det(A \circ B) \geqslant \det(A) \prod_{i=1}^{N} b_{ii} \geqslant \det(A)\det(B). \tag{6.6}$$

Note that the second inequality here follows as a consequence of the Hadamard inequality in (6.3). To make use of this, let $A = \Gamma_2$ and $B = \Pi$. If equation (6.4) is satisfied, then $A \circ B = \Gamma_1$. From this and the first inequality in (6.6) we see that

$$\det(\Gamma_1) = \det(\Gamma_2 \circ \Pi) \geqslant \det(\Gamma_2) \prod_{i=1}^{N} \pi_{ii} \tag{6.7}$$

where π_{ii} is the ii element of Π. The diagonal elements of the Gram matrices Γ_1 and Γ_2 are all equal to 1. From this and equation (6.4), it follows that the π_{ii} must also all be equal to 1, as must be their product. We therefore obtain inequality (6.5) and this completes the proof.

An elementary, but illustrative example of this theorem is the simple case of $N = 2$. Let the $|\psi_j^1\rangle$ be the initial pair of states and $|\psi_j^2\rangle$ be the final pair. Then

$$\det(\Gamma_r) = 1 - |\langle\psi_1^r|\psi_2^r\rangle|^2. \tag{6.8}$$

Combining this with inequality (6.5), we see that it is equivalent to the overlap of $|\psi_1^1\rangle$ and $|\psi_2^1\rangle$ being no greater than the overlap of $|\psi_1^2\rangle$ and $|\psi_2^2\rangle$. So, in the simple case of $N = 2$, inequality (6.5) expresses the non-decreasing nature of the overlap of the pair of states.

Let us now carry out a corresponding analysis for the difference detection probability in equation (6.2). Here, $P_{\text{diff}}(\Psi)$ is the probability of successfully confirming that the states of all N particles are different. Could this be a suitable measure of the distinguishability of the single-particle states?

This question is, of course, equivalent to asking if the corresponding minimum probability of obtaining an inconclusive result for this measurement, $P_?(\Psi)$ in equation (4.10), and

therefore the Gram matrix permanent, is a suitable measure of the indistinguishability of the single-particle states. For reasons that will become apparent, it will turn out to be more convenient to investigate the matter from this perspective. If the Gram matrix permanent is to be a suitable measure of the indistinguishability of a set of states, then intuitively it should take its minimum value when the states are orthogonal. To establish this, it would be helpful if there existed an inequality for the permanent which could play a role analogous to that played by Hadamard's inequality in our discussion of the determinant. Such an inequality exists and it is known as Marcus's inequality [24]. For any positive $N \times N$ matrix $A = (a_{ij})$, the following inequality holds:

$$\mathrm{per}(A) \geqslant \prod_{i=1}^{N} a_{ii}. \tag{6.9}$$

This is the permanental analogue of the Hadamard inequality (6.3) and, again, the equality is satisfied if and only A is diagonal. If $A = \Gamma$, then we have $\mathrm{per}(\Gamma) \geqslant 1$, where the equality is attained only in the case where the off-diagonal elements of Γ vanish, i.e. when the single-particle states are orthogonal. This confirms that $\mathrm{per}(\Gamma)$, as an indistinguishability measure, correctly identifies orthogonal states as the least indistinguishable, or equivalently, most distinguishable states. Furthermore, as we saw in section 4, $\mathrm{per}(\Gamma)$ takes its maximum value of $N!$ if and only if the states are all identical. So, it correctly identifies a set of identical states as having minimum (zero) distinguishability. We saw in our discussion of the Gram matrix determinant that, for a particular state discrimination strategy, it is not, in general, necessary for a set of states to be identical to have zero distinguishability. However, this condition is unarguably sufficient.

If the Gram matrix permanent is to be a suitable indistinguishability measure, then, under deterministic transformations of one set of pure states into another, it must display the opposite kind of behaviour to the Gram matrix determinant, i.e. it must be non-decreasing under any such transformation. That is, whenever equation (6.4) holds, we require that

$$\mathrm{per}(\Gamma_1) \leqslant \mathrm{per}(\Gamma_2). \tag{6.10}$$

It could be proven that the Gram matrix permanent is non-decreasing under such transformations if there existed a suitable permanental analogue of Oppenheim's inequality. Unfortunately, whether or not the required inequality is true in general is, at the time of writing, an open question.

The mathematical problem of finding a permanental analogue of Oppenheim's inequality has an interesting history. The first step in this direction was made by Chollet [25], who conjectured that for any two $N \times N$ positive matrices A and B,

$$\mathrm{per}(A \circ B) \leqslant \mathrm{per}(A)\,\mathrm{per}(B). \tag{6.11}$$

Subsequently, Bapat and Sunder [26] made a stronger conjecture than Chollet. They proposed that for any two positive $N \times N$ matrices A and B,

$$\mathrm{per}(A \circ B) \leqslant \mathrm{per}(A) \prod_{i=1}^{N} b_{ii} \leqslant \mathrm{per}(A)\,\mathrm{per}(B) \tag{6.12}$$

where the second inequality here is easily seen to hold as a consequence of Marcus's inequality (6.9). As a consequence of this, the Bapat–Sunder conjecture is equivalent to

$$\mathrm{per}(A \circ B) \leqslant \mathrm{per}(A) \prod_{i=1}^{N} b_{ii}. \tag{6.13}$$

Bapat and Sunder have subsequently shown that this inequality is satisfied by all 2×2 and 3×3 positive matrices [27]. The weaker conjecture (6.11) of Chollet was also proven for such matrices independently by Gregorac and Hentzel [28]. Furthermore, Bapat and Sunder [27] and Grone and Merris [29] have discovered interesting connections between the Bapat–Sunder conjecture and other conjectured identities relating to permanents.

Here, we will show that if the Bapat–Sunder conjecture is indeed true for all $N \times N$ positive matrices A and B, then inequality (6.10) always holds. Assuming the general validity of this conjecture, let us make the same substitutions as before, that is, $A = \Gamma_2$ and $B = \Pi$. If equation (6.4) is true, then

$$\mathrm{per}(\Gamma_1) = \mathrm{per}(\Gamma_2 \circ \Pi) \leqslant \mathrm{per}(\Gamma_2) \prod_{i=1}^{N} \pi_{ii} \tag{6.14}$$

where π_{ii} is the ii element of Π. The diagonal elements of the Gram matrices are all equal to 1. From equation (6.4), it follows that the π_{ii} must also all be equal to 1, as must be their product, which leads to (6.10).

For the simple case of $N = 2$, we have

$$\mathrm{per}(\Gamma_r) = 1 + \left|\left\langle \psi_1^r \middle| \psi_2^r \right\rangle\right|^2. \tag{6.15}$$

Comparing this with equation (6.8), we see that the non-decreasing nature of the permanent of the Gram matrix is, like the non-increasing nature of its determinant, equivalent to the non-decreasing of the overlap of a pair of pure states.

For $N > 3$ states, it is clear that this argument rests on whether or not the Bapat–Sunder conjecture is true. The fact that these authors proved their conjecture for all $N \times N$ positive matrices where $N \leqslant 3$ implies that the Gram matrix permanent is indeed non-decreasing under pure$\rightarrow$pure state set transformations for $N \leqslant 3$. Furthermore, it can be shown that the validity of (6.10) for every pair of $N \times N$ Gram matrices satisfying equation (6.4) is actually equivalent to the ostensibly more general Bapat–Sunder conjecture [30]. In other words, the question of whether or not the probability of a conclusive result for the universal strategy for finding at least two states to be different is a distinguishability monotone is fully equivalent to a conjectured property of $N \times N$ matrices which has been resolved affirmatively for $N \leqslant 3$ but is open for $N > 3$.

One might think that (6.10) could be established using the properties of general CPLTP maps, such as, for example, their contractivity, rather than focusing on the particular properties of transformations between sets of pure states. It would be possible to prove (6.10) if, under a general CPLTP map $\mathcal{E}$ applied separately to N systems prepared in an arbitrary initial product state, the expectation value of the symmetric subspace projector was non-decreasing, that is

$$\mathrm{Tr}[(\mathcal{E}(\rho_1) \otimes \cdots \otimes \mathcal{E}(\rho_N)) P(\mathcal{H}_{\mathrm{sym}})] \geqslant \mathrm{Tr}[(\rho_1 \otimes \cdots \otimes \rho_N) P(\mathcal{H}_{\mathrm{sym}})] \tag{6.16}$$

where the $\rho_j = |\psi_j\rangle\langle\psi_j|$ are arbitrary pure states in $\mathcal{H}$. However, this is not the case. As a counterexample, let $\mathcal{E}$ be the CPLTP map which transforms every density operator on $\mathcal{H}$ into the maximally mixed state and let each $|\psi_j\rangle = |\psi\rangle$ for some $|\psi\rangle \in \mathcal{H}$. Then, clearly the N-particle initial state lies in the symmetric subspace and so the rhs of (6.16) is equal to 1. However, the lhs is equal to $D(\mathcal{H}_{\mathrm{sym}})/D(\mathcal{H})^N < 1$ for $D(\mathcal{H}) > 1$. So, for this operation $\mathcal{E}$, the expectation value of the symmetric subspace projector can decrease, disproving the validity of (6.16) in general. As a consequence of this, we see that the expectation value of the symmetric subspace projector is not a suitable indistinguishability measure for mixed states.

7. Unambiguous overlap filtering

So far in this paper, we have been discussing the issue of unambiguous state comparison, which involves determining whether or not a number of quantum systems have been prepared in the same state. We have focused on the simple case where all possible states of the individual systems are pure. When this is so, we can view state comparison as the determination of whether or not the overlaps of the states of pairs of systems are equal to 1.

This suggests the following generalization: we would like to determine, unambiguously, whether or not the overlap of the states of a pair of systems is equal to some ω, where ω is a real parameter that can take any fixed value between 0 and 1. We shall term this procedure unambiguous overlap filtering. The problem of measuring overlaps, and relatedly, fidelities has recently received some attention. Winter [31] has shown that there is no operational procedure for measuring the fidelity, or equivalently the overlap, of the states of two quantum systems. More recently, Ekert *et al* [32] have discussed interesting relationships between fidelity/overlap estimation and a number of other important tasks in quantum information processing.

Consider two particles, 1 and 2, each with a copy of the $D(\mathcal{H})$-dimensional Hilbert space $\mathcal{H}$. The Hilbert space of the total system is $\mathcal{H}_{\rm tot} = \mathcal{H} \otimes \mathcal{H}$. Throughout this section, the left and right tensor factors will refer to particles 1 and 2, respectively. Let 1 and 2 be initially prepared in the pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. These states are taken to be completely arbitrary and unknown pure states in $\mathcal{H}$. In unambiguous state comparison, our aim is to determine with zero probability of error but with the possibility of an inconclusive result whether or not the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are identical. In other words, we aim to determine whether or not the overlap of these states, $|\langle\psi_1|\psi_2\rangle|$, is equal to 1. In unambiguous overlap filtering, our aim is to decide unambiguously whether or not $|\langle\psi_1|\psi_2\rangle| = \omega$ for some arbitrary fixed $\omega \in [0, 1]$.

We would like a measurement whose result tells us, unambiguously, that the overlap is, or is not equal to ω, with some non-zero probability. We then require a filter which takes as its inputs particles 1 and 2 prepared in the unknown pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. The measurement is required to unambiguously distinguish the cases $|\langle\psi_1|\psi_2\rangle| = \omega$ and $|\langle\psi_1|\psi_2\rangle| \neq \omega$.

This measurement will have three possible outcomes: 'yes', which signals that the overlap is equal to ω, 'no', which signals that the overlap is not equal to ω, and '?', which is an inconclusive result. These requirements imply that, formally, the measurement will be described by a three-element POVM. The three elements are $E_{\rm yes}(\omega)$, $E_{\rm no}(\omega)$ and $E_?(\omega)$. These are positive operators on the space $\mathcal{H}_{\rm tot}$, and the unambiguous nature of their outcomes implies the conditions

$$\langle\psi_1| \otimes \langle\psi_2|E_{\rm yes}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle \neq 0 \Rightarrow |\langle\psi_2|\psi_1\rangle| = \omega, \tag{7.1}$$

$$\langle\psi_1| \otimes \langle\psi_2|E_{\rm no}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle \neq 0 \Rightarrow |\langle\psi_2|\psi_1\rangle| \neq \omega. \tag{7.2}$$

They must also resolve the identity

$$E_{\rm yes}(\omega) + E_{\rm no}(\omega) + E_?(\omega) = 1_{\rm tot}. \tag{7.3}$$

If we obtain the result 'yes', then we know for sure that $|\langle\psi_2|\psi_1\rangle| = \omega$. If, on the other hand, we obtain the result 'no', then we know for sure that $|\langle\psi_2|\psi_1\rangle| \neq \omega$.

We will now prove that, if we require equations (7.1) and (7.2) to hold for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$, then

$$E_{\rm yes}(\omega) = 0 \,\forall \omega \in [0, 1], \tag{7.4}$$

$$E_{\rm no}(\omega) \neq 0 \Rightarrow \omega = 1. \tag{7.5}$$

To prove equation (7.4), let $\mathcal{H}_{12}$ be a two-dimensional subspace of $\mathcal{H}$ which contains both $|\psi_1\rangle$ and $|\psi_2\rangle$. When $|\psi_1\rangle$ and $|\psi_2\rangle$ are not identical, $\mathcal{H}_{12}$ is clearly uniquely defined as the subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$. Let $\{|x_1\rangle, |x_2\rangle\}$ and $\{|y_1\rangle, |y_2\rangle\}$ be, possibly non-orthogonal, basis sets for $\mathcal{H}_{12}$ such that

$$|\langle x_k|y_l\rangle| \neq \omega. \tag{7.6}$$

Such basis sets can always be constructed. We may write

$$|\psi_j\rangle = \sum_{k=1}^{2} a_{jk}|x_k\rangle = \sum_{l=1}^{2} b_{jl}|y_l\rangle \tag{7.7}$$

where $j = 1, 2$. Suppose now that $|\langle\psi_2|\psi_1\rangle| = \omega$ and calculate

$$\langle\psi_1| \otimes \langle\psi_2|E_{\rm yes}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle = \sum_{k,k',l,l'=1}^{2} a^*_{1k'} b^*_{2l'} a_{1k} b_{2l} \langle x_{k'}| \otimes \langle y_{l'}|E_{\rm yes}(\omega)|x_k\rangle \otimes |y_l\rangle. \tag{7.8}$$

As we shall now see, all matrix elements of $E_{\rm yes}(\omega)$ in this sum are vanishing. To prove this, we use the Cauchy–Schwarz inequality, which gives

$$\begin{aligned} &|\langle x_{k'}| \otimes \langle y_{l'}|E_{\rm yes}(\omega)|x_k\rangle \otimes |y_l\rangle|^2 \\ &\qquad \leqslant \langle x_{k'}| \otimes \langle y_{l'}|E_{\rm yes}(\omega)|x_{k'}\rangle \otimes |y_{l'}\rangle\langle x_k| \otimes \langle y_l|E_{\rm yes}(\omega)|x_k\rangle \otimes |y_l\rangle = 0 \end{aligned} \tag{7.9}$$

as a consequence of combining equation (7.1) with equation (7.6). We then conclude that $\langle\psi_1| \otimes \langle\psi_2|E_{\rm yes}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle = 0$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. This can be used to prove that $E_{\rm yes}(\omega) = 0$ in the following way: we calculate the trace of $E_{\rm yes}(\omega)$ in a product basis and from equations (7.8) and (7.9) it follows that $\mathrm{Tr}[E_{\rm yes}(\omega)] = 0$. The only positive operator with zero trace is the zero operator, so it follows that $E_{\rm yes}(\omega) = 0$.

To prove equation (7.5), we note that if $\omega < 1$ then $\mathcal{H}_{12}$ will have a possibly non-orthogonal basis set $\{|z_k\rangle\}$ with the property

$$|\langle\psi_1|z_k\rangle| = \omega. \tag{7.10}$$

This condition can always be satisfied for $\omega < 1$. For example, we may take $|z_1\rangle = \omega|\psi_1\rangle + \sqrt{1-\omega^2}\big|\psi_1^{\perp}\big\rangle$ and $|z_2\rangle = \omega|\psi_1\rangle - \sqrt{1-\omega^2}\big|\psi_1^{\perp}\big\rangle$ where $\big\langle\psi_1\big|\psi_1^{\perp}\big\rangle = 0$ and $\big|\psi_1^{\perp}\big\rangle$ is the state vector in $\mathcal{H}_{12}$ orthogonal to $|\psi_1\rangle$. Note that equation (7.10) cannot be satisfied if $\omega = 1$ since, when this is the case, the states $|z_1\rangle$ and $|z_2\rangle$ will differ at most by a phase and so they cannot be a basis for $\mathcal{H}_{12}$. In the case of $\omega < 1$, we may then write

$$|\psi_2\rangle = \sum_{k=1}^{2} c_k|z_k\rangle. \tag{7.11}$$

Suppose now that $|\langle\psi_2|\psi_1\rangle| \neq \omega$ and calculate

$$\langle\psi_1| \otimes \langle\psi_2|E_{\rm no}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle = \sum_{k,k'=1}^{2} c^*_{k'} c_k \langle\psi_1| \otimes \langle z_{k'}|E_{\rm no}(\omega)|\psi_1\rangle \otimes |z_k\rangle. \tag{7.12}$$

As we shall now see, all terms in this sum are vanishing. To prove this, we again use the Cauchy–Schwarz inequality

$$\begin{aligned} &|\langle\psi_1| \otimes \langle z_{k'}|E_{\rm no}(\omega)|\psi_1\rangle \otimes |z_k\rangle|^2 \\ &\qquad \leqslant \langle\psi_1| \otimes \langle z_{k'}|E_{\rm no}(\omega)|\psi_1\rangle \otimes |z_{k'}\rangle\langle\psi_1| \otimes \langle z_k|E_{\rm no}(\omega)|\psi_1\rangle \otimes |z_k\rangle = 0 \end{aligned} \tag{7.13}$$

where the equality follows from combining equation (7.2) with equation (7.10). So, $\langle\psi_1| \otimes \langle\psi_2|E_{\rm no}(\omega)|\psi_1\rangle \otimes |\psi_2\rangle = 0$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. In the same way as with $E_{\rm yes}(\omega)$, we can use this to prove that for $\omega \in [0, 1)$, $E_{\rm no}(\omega) = 0$. To do so, we calculate the trace of

$E_{\mathrm{no}}(\omega)$ in a product basis and use equations (7.12) and (7.13) to obtain $\mathrm{Tr}[E_{\mathrm{no}}(\omega)] = 0$. This, together with the positivity of $E_{\mathrm{no}}(\omega)$ implies that $E_{\mathrm{no}}(\omega) = 0$.

The upshot of this is that, for any given value of the overlap ω, it is impossible to unambiguously confirm, always with non-zero probability, that for two particles prepared in arbitrary, unknown pure states with overlap ω, that ω is indeed the value of their overlap. Furthermore, when their overlap is not equal to ω, it is also impossible to confirm this, again with non-zero probability for all states, unless $\omega = 1$, which corresponds to the case of the two states being identical. So, the only kind of universal unambiguous overlap filtering which is possible is universal difference detection.

8. Discussion

In this paper, we have investigated the possibility of unambiguous state comparison. As with many other operations on quantum systems, such as state discrimination and cloning, the nature of quantum operations and measurements imposes restrictions upon our ability to carry this out. We began by examining the possibility of determining when the states of N particles are all identical. If we wish to unambiguously determine when this is the case, and always with non-zero probability when it is true, then the possible states of the individual particles must form a linearly independent set. This result adds to the growing number of probabilistic quantum operations which are only possible on linearly independent sets. Among these are unambiguous state discrimination [13], probabilistic cloning [33] and unambiguous discrimination among unitary operators [34].

Although it is impossible to unambiguously confirm identicality if these single-particle states are linearly dependent, there remains the possibility of confirming unambiguously that they are not all identical. We have referred to this as difference detection and found that it can be done for all N-particle product states with a single measurement. This is simply a collective, von Neumann measurement on the N-particle system which distinguishes between the symmetric subspace and its orthogonal complement, the asymmetric subspace. We found that it is actually optimal in situations where all product states of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$ are possible. The experimental implementation of this measurement is discussed in [8], where the comparison of photon polarization states is considered. The emphasis is on the realization of this measurement using only linear optical elements. It is shown that this can be achieved perfectly for a pair of photons. It is also shown here that unambiguous difference detection can be accomplished for more than two photons using linear optical techniques, although possibly with less than the theoretical minimum probability of inconclusive results in equation (4.24).

We also considered the problem of determining unambiguously when all N particles are in different states. For this to be possible, we found that a necessary and sufficient condition is that the actual states of the N particles form a linearly independent set. We derived a universal measurement which can be used to confirm that all N particles are in different states, whenever this linear independence condition is satisfied. This measurement is a projective measurement which distinguishes between the antisymmetric subspace and its orthogonal complement, the non-antisymmetric subspace. We showed that this measurement is optimal when all N-particle product states are possible.

For a particular N-particle product state, the conclusive result probabilities for these universal measurements, given by equations (6.1) and (6.2), depend on the states of the individual particles. We considered the possibility that these probabilities can serve as measures of the distinguishability of these single-particle states. We found that they do indeed possess many of the attributes required of a distinguishability measure. The probabilities in equations (6.1) and (6.2) both maximize for orthogonal states. They also minimize for sets of

linearly dependent and identical states, respectively. While identical sets of states are always considered to have minimum (zero) distinguishability, linearly dependent sets also have this property in some contexts, e.g. in unambiguous state discrimination.

One additional requirement of a distinguishability measure is that it should be non-increasing under a deterministic quantum operation. We confined our attention to deterministic operations which transform one set of pure states into another, since all our analysis of state comparison is concerned with such states. We found that the conclusive result probability in equation (6.1) is indeed non-increasing under such transformations. The situation is more complicated for the success probability in equation (6.2). While we were able to confirm that it is non-increasing for sets of two and three states, the question of whether or not this is true for all N is equivalent to an open conjecture relating to permanents and Hadamard products of matrices. The investigation and resolution of this matter is of considerable importance in its own right. However, our demonstration that the outcome will have implications for quantum measurement and information theory gives a further incentive to pursue it. As it happens, in the simplest case of just two states, the non-increasing of these probabilities is equivalent to the non-decreasing of the overlap of the pair of states.

We concluded with the exploration of a potential generalization of unambiguous state comparison which we termed unambiguous overlap filtering. When comparing the states of two quantum systems, if these states are pure, then the comparison of these states is equivalent to determining whether or not their overlap is equal to 1. This suggests the following generalization: can we determine, unambiguously, whether or not the (pure) states of two quantum systems have overlap equal to ω where ω may take any fixed value between 0 and 1? What we have found is that it is impossible to unambiguously confirm that two unknown pure states have overlap ω for any value of ω. This generalises our earlier result that unambiguous identicality confirmation for a pair of unknown pure states is impossible [2] (which is explained in this paper by the fact that unambiguous identicality confirmation is possible only for linearly independent sets). More surprising, however, was our discovery that unambiguous confirmation that the overlap is not equal to ω for unknown pure states is possible only if $\omega = 1$. This, of course, corresponds to difference detection. So, the only kind of unambiguous overlap filtering for unknown pure states which is physically possible is unambiguous difference detection.

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